

MORITA THEORY FOR HOPF ALGEBROIDS AND PRESHEAVES OF GROUPOIDS

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ABSTRACT. Comodules over Hopf algebroids are of central importance in algebraic topology. It is well-known that a Hopf algebroid is the same thing as a presheaf of groupoids on \mathbf{Aff} , the opposite category of commutative rings. We show in this paper that a comodule is the same thing as a quasi-coherent sheaf over this presheaf of groupoids. We prove the general theorem that internal equivalences of presheaves of groupoids with respect to a Grothendieck topology \mathcal{T} on \mathbf{Aff} give rise to equivalences of categories of sheaves in that topology. We then show using faithfully flat descent that an internal equivalence in the flat topology gives rise to an equivalence of categories of quasi-coherent sheaves. The corresponding statement for Hopf algebroids is that weakly equivalent Hopf algebroids have equivalent categories of comodules. We apply this to formal group laws, where we get considerable generalizations of the Miller-Ravenel [MR77] and Hovey-Sadofsky [HS99] change of rings theorems in algebraic topology.

INTRODUCTION

A commutative Hopf algebra is a (commutative) ring A together with a lift of the functor $\mathrm{Spec} A: \mathbf{Rings} \rightarrow \mathbf{Set}$ to a functor $\mathbf{Rings} \rightarrow \mathbf{Groups}$. Here \mathbf{Rings} is the category of commutative rings with unity, \mathbf{Set} is the category of sets, \mathbf{Groups} is the category of groups, and $(\mathrm{Spec} A)(R) = \mathbf{Rings}(A, R)$. So a Hopf algebra is the same thing as an affine algebraic group scheme, or a representable presheaf of groups on \mathbf{Aff} , the opposite category of \mathbf{Rings} . In the same way, a Hopf algebroid (A, Γ) is an affine algebraic groupoid scheme, or a representable presheaf of groupoids $(\mathrm{Spec} A, \mathrm{Spec} \Gamma)$ on \mathbf{Aff} . Here, given a ring R , $\mathrm{Spec} A(R)$ is the set of objects of the groupoid corresponding to R , and $\mathrm{Spec} \Gamma(R)$ is the set of morphisms of that groupoid.

Hopf algebroids are very important in algebraic topology, because for many important homology theories E , the ring of stable co-operations E_*E is a (graded) Hopf algebroid over E_* but not a Hopf algebra. In particular, this is true for complex cobordism MU and complex K -theory. In this case, E_*X is a (graded) comodule over the Hopf algebroid E_*E .

Of course, not all schemes are affine. One of the essential contributions of Grothendieck was the realization that it is necessary to study all schemes even if one is only interested in affine schemes. In the same way, to understand Hopf algebroids, one should study more general groupoid schemes.

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One of the difficulties is that the standard approach to schemes, involving covers by open affine subschemes, is not the right one for the algebraic topology setting. Instead, it is better to use the functorial approach hinted at above in our definition of $\mathrm{Spec} A$. This approach is well-known in algebraic geometry [DG70]. It was introduced to algebraic topology by Hopkins and Neil Strickland. Strickland has written an excellent exposition of this point of view in [Str99]. In this approach, we study arbitrary presheaves of sets (or groupoids) on \mathbf{Aff} .

Demazure and Gabriel [DG70] show that the category of A -modules is equivalent to the category of quasi-coherent sheaves over the presheaf of sets $\mathrm{Spec} A$ on \mathbf{Aff} . Our first goal in this paper is to extend this theorem as follows. Let \mathcal{T} denote a Grothendieck topology on \mathbf{Aff} , and let $\mathbf{Aff}_{\mathcal{T}}$ denote the resulting site (we put a cardinality restriction on rings to make \mathbf{Aff} a small category). Given a presheaf of groupoids (X_0, X_1) on \mathbf{Aff} , we define the category $\mathbf{Sh}_{(X_0, X_1)}^{\mathcal{T}}$ of sheaves over (X_0, X_1) with respect to \mathcal{T} and we define the category $\mathbf{Sh}_{(X_0, X_1)}^{\mathrm{qc}}$ of quasi-coherent sheaves over (X_0, X_1) . Our first main result is then the following theorem, proved as Theorem 2.2.

Theorem A. *Suppose (A, Γ) is a Hopf algebroid. Then there is an equivalence of categories between Γ -comodules and quasi-coherent sheaves over $(\mathrm{Spec} A, \mathrm{Spec} \Gamma)$.*

There is a natural notion of an internal equivalence of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$, studied by Joyal and Tierney [JT91] and other authors as well. A map $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ of presheaves of groupoids is an internal equivalence with respect to \mathcal{T} if $\Phi(R)$ is fully faithful for all R and if Φ is essentially surjective in a sheaf-theoretic sense, related to \mathcal{T} . This is really the natural notion of internal equivalence for *sheaves* of groupoids on $\mathbf{Aff}_{\mathcal{T}}$; there is a more general notion appropriate for presheaves, introduced by Hollander [Hol01], but we do not need it.

Our second main result is that the category of sheaves is invariant under internal equivalence. The following theorem is proved as Theorem 3.2.

Theorem B. *Suppose $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is an internal equivalence of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$. Then $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\mathcal{T}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\mathcal{T}}$ is an equivalence of categories.*

What we really care about is the category of quasi-coherent sheaves. Faithfully flat descent shows that a quasi-coherent sheaf is a sheaf in the flat topology on \mathbf{Aff} . This is often called the fpqc topology; in it, a cover of a ring R is a finite family $\{R \rightarrow S_i\}$ of flat extensions of R such that $\prod S_i$ is faithfully flat over R . A strengthening of faithfully flat descent then leads to the following theorem, proved as Theorem 4.5.

Theorem C. *Suppose $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is an internal equivalence of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$, where \mathcal{T} is the flat topology. Then $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\mathrm{qc}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\mathrm{qc}}$ is an equivalence of categories.*

In order to apply this theorem to Hopf algebroids, we need to characterize those maps of Hopf algebroids that induce internal equivalences in the flat topology of the corresponding presheaves of groupoids. The following theorem is proved as Theorem 5.5.

Theorem D. *Suppose $f = (f_0, f_1): (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of Hopf algebroids. Then $f^*: (\mathrm{Spec} B, \mathrm{Spec} \Sigma) \rightarrow (\mathrm{Spec} A, \mathrm{Spec} \Gamma)$ is an internal equivalence in the flat*

topology if and only if

$$\eta_L \otimes f_1 \otimes \eta_R: B \otimes_A \Gamma \otimes_A B \rightarrow \Sigma$$

is an isomorphism and there is a ring map $g: B \otimes_A \Gamma \rightarrow C$ such that $g(f_0 \otimes \eta_R)$ exhibits C as a faithfully flat extension of A .

This condition has appeared before, in [Hop95] and [HS99]. We point out that if we used the more general notion of internal equivalence mentioned above, Theorem D would remain unchanged, since $\text{Spec } A$ is already a sheaf in the flat topology by faithfully flat descent.

Finally, we apply our results to the Hopf algebroids relevant to algebraic topology. The following theorem is proved as Theorem 6.2 (and the terminology is defined in Section 6).

Theorem E. *Fix a prime p and an integer $n > 0$. Let (A, Γ) denote the Hopf algebroid $(v_n^{-1}BP_*/I_n, v_n^{-1}BP_*BP/I_n)$. Suppose B is a ring equipped with a homogeneous p -typical formal group law of strict height n , classified by $f: A \rightarrow B$. Then the functor that takes an (A, Γ) -comodule M to $B \otimes_A M$ defines an equivalence of categories from graded (A, Γ) -comodules to graded $(B, B \otimes_A \Gamma \otimes_A B)$ -comodules.*

As an immediate corollary, we recover a strengthening of the change of rings theorem of [HS99], which itself is a strengthening of the well-known Miller-Ravenel change of rings theorem [MR77]. The precise change of rings theorem is proved below.

Theorem F. *Let p be a prime and $m \geq n > 0$ be integers. Suppose M and N are BP_*BP -comodules such that v_n acts isomorphically on N . If either M is finitely presented, or if $N = v_n^{-1}N'$ where N' is finitely presented and I_n -nilpotent, then*

$$Ext_{BP_*BP}^{**}(M, N) \cong Ext_{E(m)_*E(m)}^{**}(E(m)_* \otimes_{BP_*} M, E(m)_* \otimes_{BP_*} N).$$

This theorem implies that the chromatic spectral sequence based on $E(m)$ is the truncation of the chromatic spectral sequence based on BP consisting of the first $n + 1$ columns, as pointed out in [HS99, Remark 5.2].

There are several ways in which the results in this paper might be generalized. Most substantively, we do not recover the Morava change of rings theorem [Mor85] from our result. The Morava change of rings theorem is about complete comodules over a complete Hopf algebroid, so one would need to account for the topology in some way. Secondly, our results will probably hold if we replace **Aff** by the opposite category of rings in some topos, as suggested by Rick Jardine. In fact, we already have to replace **Aff** by the opposite category of graded rings in order to cope with the graded Hopf algebroids that arise in algebraic topology. Lastly, there is the aforementioned generalization of the notion of internal equivalence, due to Hollander [Hol01]. In this generalization, one would replace “faithful” by “sheaf-theoretically faithful” and “full” by “sheaf-theoretically full”. We are confident our results will hold for this generalization, but we would not get any new examples of equivalences of categories of comodules. Nevertheless, this generalization might be useful in other circumstances.

This paper arose from trying to understand comments of Mike Hopkins, and I thank him deeply for sharing his insights. The one-line summary of this paper is “The category of comodules over a Hopf algebroid only depends on the associated stack”, and the author first heard this summary from Hopkins. It is certain that

Hopkins has proved some of the theorems in this paper. As far as I know, however, Hopkins approached these theorems by using stacks, which I have completely avoided. In particular, my definition of sheaves and quasi-coherent sheaves over presheaves of groupoids is quite different from the definition I have heard from Hopkins, though the two definitions are presumably equivalent.

The author would also like to thank Dan Christensen and Rick Jardine, both of whom thought that the original version of this paper, dealing as it did with only quasi-coherent sheaves, was much too specific and must be a corollary of a simpler, more general theorem.

NOTATION

We compile the notations and conventions we use in this paper. All rings are assumed commutative, and of cardinality less than some fixed infinite cardinal κ . **Rings** denotes the category of such rings, and **Aff** denotes its opposite category. We think of **Aff** as the category of representable functors $\text{Spec } A: \mathbf{Rings} \rightarrow \mathbf{Set}$, where $(\text{Spec } A)(R) = \mathbf{Rings}(A, R)$. We will also want to consider **Rings**_{*}, the category of graded rings (of cardinality less than κ) that are commutative in the graded sense, and its opposite category **Aff**_{*}.

If $x, y: A \rightarrow R$ are ring homomorphisms, the symbol ${}_x R_y$ denotes R with its A -bimodule structure, where A acts on the left through x and on the right through y . This is especially useful for the tensor product; the symbol $R_x \otimes_A {}_y S$ indicates the bimodule tensor product, where A acts on the right on R via x and on the left on S via y . We use this same notation in the graded case as well, where x and y are tacitly assumed to preserve the grading and the tensor product is the graded tensor product.

The symbols (A, Γ) and (B, Σ) denote (possibly graded) Hopf algebroids. We follow the notation of [Rav86, Appendix 1] for the structure maps of a Hopf algebroid. So we have the counit $\epsilon: \Gamma \rightarrow A$, the left and right units $\eta_L, \eta_R: A \rightarrow \Gamma$, the diagonal $\Delta: \Gamma \rightarrow \Gamma_{\eta_R} \otimes_A \Gamma_{\eta_L}$, and the conjugation $c: {}_{\eta_L} \Gamma_{\eta_R} \rightarrow {}_{\eta_R} \Gamma_{\eta_L}$.

Capital letters at the end of the alphabet, such as X , Y , and Z , will denote functors from **Rings** to **Set**, or functors from **Rings**_{*} to **Set** in the graded case.

The symbol $Y_f \times_X {}_g Z$ will denote the pullback of the diagram $Y \xrightarrow{f} X \xleftarrow{g} Z$.

The symbols (X_0, X_1) and (Y_0, Y_1) will denote functors from **Rings** (or **Rings**_{*}) to **Gpds**, the category of small groupoids. Here $X_0(R)$ is the object set of the groupoid corresponding to R , and $X_1(R)$ is the morphism set of that groupoid.

There are structure maps

$$\begin{aligned} \text{id}: X_0 &\rightarrow X_1 \\ \text{dom, codom}: X_1 &\rightarrow X_0 \\ \circ: (X_1)_{\text{dom}} \times_{X_0} (X_1)_{\text{codom}} &\rightarrow X_1 \\ \text{inv}: X_1 &\rightarrow X_1 \end{aligned}$$

satisfying the relations necessary to make $(X_0(R), X_1(R))$ a groupoid.

1. SHEAVES OVER FUNCTORS

The object of this section is to define the notion of a sheaf of modules M over a sheaf of sets X on **Aff**. We will generalize this in the next section to sheaves of modules over sheaves of groupoids (X_0, X_1) on **Aff**.

We will assume given a Grothendieck topology \mathcal{T} on \mathbf{Aff} , and denote the resulting site consisting of \mathbf{Aff} together with \mathcal{T} by $\mathbf{Aff}_{\mathcal{T}}$. For us, the two most important Grothendieck topologies on \mathbf{Aff} will be the trivial topology, where the only covers are isomorphisms, and the fpqc, or flat, topology, which will be discussed later.

Now suppose $X: \mathbf{Rings} \rightarrow \mathbf{Set}$ is a functor. We think of X as a presheaf of sets on $\mathbf{Aff}_{\mathcal{T}}$. We need to define the category of sheaves over X . We first define the overcategory $\mathbf{Aff}_{\mathcal{T}}/X$. An object of $\mathbf{Aff}_{\mathcal{T}}/X$ is a map of presheaves $\mathrm{Spec} R \xrightarrow{x} X$, and the morphisms are the commutative triangles. We call the opposite category of $\mathbf{Aff}_{\mathcal{T}}/X$ the category of *points* of X following [Str99]; it is called the category of X -models in [DG70]. A point of X is a pair (R, x) , where R is a ring and $x \in X(R)$, and a morphism from (R, x) to (S, y) is a ring homomorphism $f: R \rightarrow S$ such that $X(f)(x) = y$. We often abuse notation and write $f(x)$ for $X(f)(x)$. As an overcategory, $\mathbf{Aff}_{\mathcal{T}}/X$ inherits the Grothendieck topology \mathcal{T} . A cover of (R, x) is a family $\{(R, x) \rightarrow (S_i, x_i)\}$ such that $\{R \rightarrow S_i\}$ is a cover of R . The category $\mathbf{Aff}_{\mathcal{T}}/X$ also comes equipped with a structure presheaf $\mathcal{O}: (\mathbf{Aff}_{\mathcal{T}}/X)^{\mathrm{op}} \rightarrow \mathbf{Rings}$, where $\mathcal{O}(R, x) = R$.

Definition 1.1. Suppose $X: \mathbf{Rings} \rightarrow \mathbf{Set}$ is a presheaf of sets on $\mathbf{Aff}_{\mathcal{T}}$. Then a *sheaf of modules over X* , often called just a *sheaf over X* , is a sheaf of \mathcal{O} -modules on $\mathbf{Aff}_{\mathcal{T}}/X$.

More concretely, a sheaf M is a functorial assignment of an R -module M_x to each point (R, x) , satisfying the sheaf condition. Functoriality means that a map $(R, x) \xrightarrow{f} (S, y)$ induces a map of R -modules $M_x \xrightarrow{\theta_M(f, x)} M_y$, where M_y is thought of as an R -module by restriction. We often abbreviate $\theta(f, x)$ to $\theta(f)$. We must have $\theta(gf) = \theta(g) \circ \theta(f)$ and $\theta(1) = 1$. The sheaf condition means that if $\{(R, x) \rightarrow (S_i, x_i)\}$ is a cover, then the diagram

$$M_x \rightarrow \prod_i M_{x_i} \rightrightarrows \prod_{jk} M_{x_{jk}}$$

is an equalizer of R -modules, where x_{jk} is the image of x in $X(S_j \otimes_R S_k)$. The maps in this diagram are all maps of R -modules.

We have an evident definition of a map of sheaves over X . To be concrete, a map $\alpha: M \rightarrow N$ of sheaves over X assigns to each point (R, x) of X a map $\alpha_x: M_x \rightarrow N_x$ of R -modules, natural in (R, x) . This gives us a category $\mathbf{Sh}_X^{\mathcal{T}}$ of sheaves over X . A map of sheaves $X \xrightarrow{\Phi} Y$ induces a functor $\Phi^*: \mathbf{Sh}_Y^{\mathcal{T}} \rightarrow \mathbf{Sh}_X^{\mathcal{T}}$. Here, if M is a sheaf over Y and (R, x) is a point of X , we define $(\Phi^*M)_x = M_{\Phi x}$.

Note that all of these definitions work perfectly well in the graded case as well. We would have a Grothendieck topology \mathcal{T} on \mathbf{Aff}_* , and a functor $X: \mathbf{Aff}_* \rightarrow \mathbf{Set}$. A point of X would be a graded ring R and a point $x \in X(R)$. A sheaf M over X would be an assignment of a graded R -module M_x to each point (R, x) of X , satisfying the functoriality and sheaf conditions.

We now consider quasi-coherent sheaves. We only need quasi-coherent sheaves in the trivial topology, so we will stick to that case. A quasi-coherent sheaf is supposed to be a sheaf that acts like a free sheaf in an appropriate sense. The salient property of the free sheaf \mathcal{O} is that, if $(R, x) \rightarrow (S, y)$ is a map of points, then $\mathcal{O}_y = S \otimes_R \mathcal{O}_x$. We therefore make the following definition.

Definition 1.2. Suppose $X: \mathbf{Rings} \rightarrow \mathbf{Set}$ is a functor. A *quasi-coherent sheaf M over X* is a sheaf over X in the trivial topology such that, given a map

$(R, x) \rightarrow (S, y)$ of points of X , the adjoint $\rho_M(f): S \otimes_R M_x \rightarrow M_y$ of $\theta_M(f)$ is an isomorphism.

This is the same definition given in [DG70] and [Str99]. We get a category $\mathbf{Sh}_X^{\text{qc}}$, which is the full subcategory of sheaves over X in the trivial topology consisting of the quasi-coherent sheaves. Given a map $\Phi: X \rightarrow Y$ of functors, $\Phi^*: \mathbf{Sh}_Y^{\mathcal{T}} \rightarrow \mathbf{Sh}_X^{\mathcal{T}}$ restricts to define $\Phi^*: \mathbf{Sh}_Y^{\text{qc}} \rightarrow \mathbf{Sh}_X^{\text{qc}}$.

The value of this definition of quasi-coherence is shown by the following lemma.

Lemma 1.3. *Suppose $A \in \mathbf{Rings}$, and let $\text{Spec } A: \mathbf{Rings} \rightarrow \mathbf{Set}$ be the representable functor $(\text{Spec } A)(R) = \mathbf{Rings}(A, R)$. Then the category of A -modules is equivalent to the category of quasi-coherent sheaves over $\text{Spec } A$. The equivalence takes an A -module M to the quasi-coherent sheaf \widetilde{M} over $\text{Spec } A$ defined by $\widetilde{M}_x = R_x \otimes_A M$ for $x: A \rightarrow R$, and its inverse takes a quasi-coherent sheaf N to its value at 1: $A \rightarrow A$.*

This lemma is due to Demazure and Gabriel [DG80, p. 61], who actually show that the category of quasi-coherent sheaves over a scheme when defined this way agrees (up to equivalence) with the usual notion of quasi-coherent sheaves on a scheme. A direct proof can be found in [Str99].

Once again, we note that Lemma 1.3 will work in the graded case as well. The definition of a quasi-coherent sheaf over a functor $X: \mathbf{Rings}_* \rightarrow \mathbf{Set}$ is similar to the ungraded case, and the same argument used to prove Lemma 1.3 shows that, if A is a graded ring, the category of quasi-coherent sheaves over $\text{Spec } A$ (now defined by $(\text{Spec } A)(R) = \mathbf{Rings}_*(A, R)$) is equivalent to the category of graded A -modules.

It will be useful later to note that, if $f: A \rightarrow B$ is a ring homomorphism and $\text{Spec } f: \text{Spec } B \rightarrow \text{Spec } A$ is the corresponding map of functors, then the induced map $(\text{Spec } f)^*: \mathbf{Sh}_{\text{Spec } A}^{\text{qc}} \rightarrow \mathbf{Sh}_{\text{Spec } B}^{\text{qc}}$ takes the A -module M to the B -module $B \otimes_A M$.

2. SHEAVES OVER GROUPOID FUNCTORS

The object of this section is to prove Theorem A, showing that a comodule over a Hopf algebroid is a special case of the more general notion of a quasi-coherent sheaf over a presheaf of groupoids. This will require us to define the notion of a sheaf M of modules over a presheaf of groupoids (X_0, X_1) on $\mathbf{Aff}_{\mathcal{T}}$.

We will consider a presheaf of groupoids (X_0, X_1) on $\mathbf{Aff}_{\mathcal{T}}$. This means that X_0 and X_1 are presheaves of sets on $\mathbf{Aff}_{\mathcal{T}}$, and that $(X_0(R), X_1(R))$ is a groupoid for all R , naturally in R . So we have structure maps as defined in the notation section. A presheaf of groupoids (X_0, X_1) is called a *sheaf of groupoids* when X_0 and X_1 are sheaves of sets on $\mathbf{Aff}_{\mathcal{T}}$; we would be happy to assume our presheaves of groupoids are in fact sheaves of groupoids, but that assumption is unnecessary.

Sheaves of groupoids have been much studied in the literature; a stack is a special kind of sheaf of groupoids, and stacks are essential in modern algebraic geometry [FC90]. The homotopy theory of sheaves of groupoids has been studied by Joyal and Tierney [JT91], Jardine [Jar00], and Hollander [Hol01].

Definition 2.1. Suppose (X_0, X_1) is a presheaf of groupoids on $\mathbf{Aff}_{\mathcal{T}}$. A *sheaf over (X_0, X_1)* is a sheaf M over X_0 together with an isomorphism $\psi: \text{dom}^* M \rightarrow \text{codom}^* M$ of sheaves over X_1 satisfying the cocycle condition. To explain the cocycle condition, note that, if α is a morphism of $X_1(R)$, ψ_α is an isomorphism of

R -modules $\psi_\alpha: M_{\text{dom } \alpha} \rightarrow M_{\text{codom } \alpha}$. The cocycle condition says that if β and α are composable morphisms, then $\psi_{\beta\alpha} = \psi_\beta \circ \psi_\alpha$. A *quasi-coherent sheaf over (X_0, X_1)* is a sheaf M over (X_0, X_1) in the trivial topology such that M is quasi-coherent as a sheaf over X_0 .

We also get a notion of a map $\tau: M \rightarrow N$ of sheaves over (X_0, X_1) . Such a map is a map of sheaves over X_0 such that the diagram

$$\begin{array}{ccc} M_{\text{dom } \alpha} & \xrightarrow{\psi_\alpha^M} & M_{\text{codom } \alpha} \\ \tau_{\text{dom } \alpha} \downarrow & & \downarrow \tau_{\text{codom } \alpha} \\ N_{\text{dom } \alpha} & \xrightarrow{\psi_\alpha^N} & N_{\text{codom } \alpha} \end{array}$$

commutes for all points (R, α) of $X_1(R)$. We then get categories $\mathbf{Sh}_{(X_0, X_1)}^\tau$ and $\mathbf{Sh}_{(X_0, X_1)}^{\text{qc}}$.

Note that a map $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ induces a functor $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^\tau \rightarrow \mathbf{Sh}_{(X_0, X_1)}^\tau$ and $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\text{qc}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\text{qc}}$. Indeed, we define $\psi_\alpha^{\Phi^* M} = \psi_{\Phi\alpha}^M$.

Also note that all of the comments above work perfectly well for presheaves of groupoids on \mathbf{Aff}_* . In this case, $\psi_\alpha: M_{\text{dom } \alpha} \rightarrow M_{\text{codom } \alpha}$ will be an isomorphism of graded R -modules.

As originally noted by Haynes Miller, a Hopf algebroid [Rav86, Appendix 1] is just a pair of commutative rings (A, Γ) such that $(\text{Spec } A, \text{Spec } \Gamma)$ is a sheaf of groupoids (in the trivial topology). The structure maps of a Hopf algebroid (listed in the notation section) are therefore dual to the structure maps of a presheaf of groupoids; for example, the diagonal $\Delta: \Gamma \rightarrow \Gamma_{\eta_R} \otimes_A \Gamma_{\eta_L}$ is dual to the composition map $(X_1)_{\text{dom}} \times_{X_0 \text{ codom}} X_1$.

It is useful to recall the composition in the groupoid $(\text{Spec } A, \text{Spec } \Gamma)(R)$ from this point of view. Suppose $\beta, \alpha: \Gamma \rightarrow R$ are ring homomorphisms with $\alpha\eta_L = x$, $\alpha\eta_R = \beta\eta_L = y$, and $\beta\eta_R = z$, so that α is a morphism from x to y and β is a morphism from y to z . The composition $\beta \circ \alpha: \Gamma \rightarrow R$ is defined to be the composite

$$\Gamma \xrightarrow{\Delta} \Gamma_{\eta_L} \Gamma_{\eta_R} \otimes_A \Gamma_{\eta_L} \Gamma_{\eta_R} \xrightarrow{\alpha \otimes \beta} {}_x R_y \otimes_A {}_y R_z \xrightarrow{\mu} {}_x R_z.$$

Just as a quasi-coherent sheaf over $\text{Spec } A$ is the same thing as a module over A , so a quasi-coherent sheaf over $(\text{Spec } A, \text{Spec } \Gamma)$ is the same thing as a comodule over (A, Γ) . The following theorem is Theorem A of the introduction.

Theorem 2.2. *Suppose (A, Γ) is a Hopf algebroid. Then there is an equivalence of categories between Γ -comodules and quasi-coherent sheaves over $(\text{Spec } A, \text{Spec } \Gamma)$.*

This theorem will also hold in the graded context: if (A, Γ) is a graded Hopf algebroid, then the category of graded Γ -comodules is equivalent to the category of quasi-coherent sheaves over the presheaf of groupoids $(\text{Spec } A, \text{Spec } \Gamma)$ on \mathbf{Aff}_* . The proof is the same as the proof below.

Proof. We first construct a functor from quasi-coherent sheaves over $(\text{Spec } A, \text{Spec } \Gamma)$ to (A, Γ) -comodules. Suppose that \widetilde{M} is a quasi-coherent sheaf over $(\text{Spec } A, \text{Spec } \Gamma)$. Then \widetilde{M} is in particular a quasi-coherent sheaf over $\text{Spec } A$, so corresponds to an

A -module M . Then if $\alpha: \Gamma \rightarrow R$ is a point of $\text{Spec } \Gamma$ defined over R , with $\alpha\eta_L = x$ and $\alpha\eta_R = y$,

$$(\text{dom}^* \widetilde{M})_\alpha = R_x \otimes_A M \text{ and } (\text{codom}^* \widetilde{M})_\alpha = R_y \otimes_A M.$$

Let us denote by $\widetilde{\psi}$ the isomorphism of sheaves $\text{dom}^* \widetilde{M} \rightarrow \text{codom}^* \widetilde{M}$. Then, $\widetilde{\psi}$ defines an isomorphism

$$\widetilde{\psi}_\alpha: R_x \otimes_A M \rightarrow R_y \otimes_A M.$$

of R -modules. Taking α to be the identity map 1 of Γ , we define $\psi: M \rightarrow \Gamma_{\eta_R} \otimes_A M$ to be the composite

$$M = A \otimes_A M \xrightarrow{\eta_L \otimes 1} \Gamma_{\eta_L} \otimes_A M \xrightarrow{\widetilde{\psi}_1} \Gamma_{\eta_R} \otimes_A M.$$

We must show that ψ is counital and coassociative. Note first that $\epsilon: \Gamma \rightarrow A$, thought of as a morphism in the groupoid $(\text{Spec } A, \text{Spec } \Gamma)(A)$, is the identity morphism of the object $1_A: A \rightarrow A$, and so in particular is idempotent. The cocycle condition implies that $\widetilde{\psi}_\epsilon$ is also idempotent, and since it is an isomorphism, it follows that $\widetilde{\psi}_\epsilon$ is the identity of M . Now, ϵ defines a map from the point $(\Gamma, 1)$ to the point (A, ϵ) of $\text{Spec } \Gamma$. Since $\widetilde{\psi}$ is a map of sheaves over $\text{Spec } \Gamma$, we conclude that

$$1 \otimes \widetilde{\psi}_1: A \otimes_\Gamma (\Gamma_{\eta_L} \otimes_A M) \rightarrow A \otimes_\Gamma (\Gamma_{\eta_R} \otimes_A M)$$

is the identity map. From this it follows easily that ψ is counital.

To see that ψ is coassociative, let $\alpha: \Gamma \rightarrow \Gamma \otimes_A \Gamma$ denote the map that takes t to $t \otimes 1$. Let β denote the map that takes t to $1 \otimes t$. Then we have

$$y\eta_R(a) = \eta_R a \otimes 1 = 1 \otimes \eta_L a = x\eta_L(a),$$

and so $\beta \circ \alpha$ makes sense. A calculation shows that $\beta \circ \alpha = \Delta$, the diagonal map. If (R, γ) is an arbitrary point of $\text{Spec } \Gamma$ with $\gamma\eta_L = x$ and $\gamma\eta_R = y$, there is a map from $(\Gamma, 1)$ to (R, γ) . Since $\widetilde{\psi}$ is a map of sheaves, we find that $\widetilde{\psi}_\gamma$ is the composite

$$R_x \otimes_A M \cong R_\gamma \otimes_\Gamma \Gamma_{\eta_L} \otimes_A M \xrightarrow{1 \otimes \widetilde{\psi}_1} R_\gamma \otimes_\Gamma \Gamma_{\eta_R} \otimes_A M \cong R_y \otimes_A M.$$

This description allows us to compute $\widetilde{\psi}_\beta$ and $\widetilde{\psi}_\alpha$, and so also their composite. We find that $\widetilde{\psi}_\beta \circ \widetilde{\psi}_\alpha$ takes $1 \otimes 1 \otimes m$ to $(1 \otimes \psi)\psi(m)$. Similarly $\widetilde{\psi}_\Delta$ takes $1 \otimes 1 \otimes m$ to $(\Delta \otimes 1)\psi(m)$. The cocycle condition forces these to be equal, and so ψ is coassociative.

We have now constructed a comodule M associated to any quasi-coherent sheaf \widetilde{M} over $(\text{Spec } A, \text{Spec } \Gamma)$. We leave to the reader the straightforward check that this is functorial.

Our next goal is to construct a functor from (A, Γ) -comodules to quasi-coherent sheaves over $(\text{Spec } A, \text{Spec } \Gamma)$. Suppose M is a Γ -comodule with structure map $\psi: M \rightarrow \Gamma_{\eta_R} \otimes_A M$. Then, in particular, M is an A -module, so there is an associated quasi-coherent sheaf \widetilde{M} over $\text{Spec } A$, defined by $\widetilde{M}_x = R_x \otimes_A M$, where $x: A \rightarrow R$ is a ring homomorphism. Given a point $\alpha: \Gamma \rightarrow R$ of $\text{Spec } \Gamma$ with $\alpha\eta_L = x$ and $\alpha\eta_R = y$, we have

$$(\text{dom}^* \widetilde{M})_x = R_x \otimes_A M \text{ and } (\text{codom}^* \widetilde{M})_x = R_y \otimes_A M$$

We define $\widetilde{\psi}: \text{dom}^* \widetilde{M} \rightarrow \text{codom}^* \widetilde{M}$ by letting $\widetilde{\psi}_\alpha$ be the composite

$$R_x \otimes_A M \xrightarrow{1 \otimes \psi} R_x \otimes_A \Gamma_{\eta_L} \otimes_A M \xrightarrow{1 \otimes \alpha \otimes 1} R_x \otimes_A x R_y \otimes_A M \xrightarrow{\mu \otimes 1} R_y \otimes_A M.$$

We leave to the reader the check that $\tilde{\psi}$ is a map of sheaves.

It remains to show that $\tilde{\psi}$ satisfies the cocycle condition and is an isomorphism. We begin with the cocycle condition. Suppose that $\alpha, \beta: \Gamma \rightarrow R$ are ring homomorphisms with $\alpha\eta_L = x$, $\alpha\eta_R = \beta\eta_L = y$, and $\beta\eta_R = z$. Consider the following commutative diagram, in which all tensor products that occur are taken over A , and $\Gamma = {}_{\eta_L}\Gamma_{\eta_R}$.

$$\begin{array}{ccccccc}
R_x \otimes M & \xrightarrow{1 \otimes \psi} & R_x \otimes \Gamma \otimes M & \xrightarrow{1 \otimes \alpha \otimes 1} & R_x \otimes {}_x R_y \otimes M & \xrightarrow{\mu \otimes 1} & R_y \otimes M \\
1 \otimes \psi \downarrow & & 1 \otimes 1 \otimes \psi \downarrow & & 1 \otimes 1 \otimes \psi \downarrow & & 1 \otimes 1 \otimes \psi \downarrow \\
R_x \otimes \Gamma \otimes M & \xrightarrow{1 \otimes \Delta \otimes 1} & R_x \otimes \Gamma \otimes \Gamma \otimes M & \xrightarrow{1 \otimes \alpha \otimes 1 \otimes 1} & R_x \otimes {}_x R_y \otimes \Gamma \otimes M & \xrightarrow{\mu \otimes 1 \otimes 1} & R_y \otimes \Gamma \otimes M \\
& & & & 1 \otimes 1 \otimes \beta \otimes 1 \downarrow & & 1 \otimes \beta \otimes 1 \downarrow \\
& & & & R_x \otimes {}_x R_y \otimes {}_y R_z \otimes M & \xrightarrow{\mu \otimes 1 \otimes 1} & R_y \otimes {}_y R_z \otimes M \\
& & & & 1 \otimes \mu \otimes 1 \downarrow & & \mu \otimes 1 \downarrow \\
& & & & R_x \otimes {}_x R_z \otimes M & \xrightarrow{\mu \otimes 1} & R_z \otimes M
\end{array}$$

The outer clockwise composite in this diagram is $\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha$, and the outer counterclockwise composite is $\tilde{\psi}_{\beta \circ \alpha}$, using the description of $\beta \circ \alpha$ given above. Thus $\tilde{\psi}$ satisfies the cocycle condition.

We must still show that $\tilde{\psi}_\alpha$ is an isomorphism for all $\alpha: \Gamma \rightarrow R$. Since $\tilde{\psi}$ satisfies the cocycle condition and α is itself an isomorphism, it suffices to show that $\tilde{\psi}_{1_x}$ is an isomorphism, where 1_x is the identity morphism of $x: A \rightarrow R$. That is, 1_x is the composite

$$\Gamma \xrightarrow{\epsilon} A \xrightarrow{x} R.$$

But one can check, using the fact that ψ is counital, that $\tilde{\psi}_{1_x}$ is the identity of $R_x \otimes_A M$. This completes the proof that \tilde{M} is a quasi-coherent sheaf over $(\text{Spec } A, \text{Spec } \Gamma)$. We leave to the reader the check that it is functorial in M .

We also leave to the reader the check that these constructions define inverse equivalences of categories. \square

Maps of Hopf algebroids $(f_0, f_1): (A, \Gamma) \rightarrow (B, \Sigma)$ are defined in [Rav86, Definition A1.1.7]; they are, of course, maps such that $\Phi = (\text{Spec } f_0, \text{Spec } f_1)$ is a map of sheaves of groupoids. According to Theorem 2.2, (f_0, f_1) will induce a map Φ^* from (A, Γ) -comodules to (B, Σ) -comodules. This map takes the Γ -comodule M to $B \otimes_A M$. In order to define the structure map of $B \otimes_A M$, recall from [Rav86, Definition A1.1.7] that the definition of a map of Hopf algebroids requires

$$\eta_L f_0 = x = f_1 \eta_L \text{ and } \eta_R f_0 = y = f_1 \eta_R.$$

We then define the structure map of $B \otimes_A M$ to be the composite

$$\begin{aligned}
B_{f_0} \otimes_A M & \xrightarrow{1 \otimes \psi} B \otimes_A {}_{\eta_L}\Gamma_{\eta_R} \otimes_A M \xrightarrow{\eta_L \otimes f_1 \otimes 1} \Delta_x \otimes_A {}_x \Delta_y \otimes_A M \\
& \xrightarrow{\mu \otimes 1} \Delta_y \otimes_A M \cong \Delta_{\eta_R} \otimes_B (B_{f_0} \otimes_A M).
\end{aligned}$$

3. INTERNAL EQUIVALENCES YIELD EQUIVALENCES

The object of this section is to prove Theorem B, showing that if $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is an *internal equivalence* of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$, then

$$\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\mathcal{T}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\mathcal{T}}$$

is an equivalence of categories. This statement essentially says that the category of sheaves is a homotopy-invariant construction.

We begin by defining an internal equivalence. Internal equivalences are the weak equivalences in the model structure on sheaves of groupoids considered by Joyal and Tierney in [JT91].

Definition 3.1. Suppose $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is a map of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$. The *essential image* of Φ is the subfunctor of Y_0 consisting of all points (R, y) of Y_0 such that there exists a point (R, x) of X_0 and a morphism $\alpha \in Y_1(R)$ from Φx to y . The *sheaf-theoretic essential image* of Φ is the subfunctor of Y_0 consisting of all points (R, y) such that there exists a cover $\{R \xrightarrow{f_i} S_i\}$ of R in the topology \mathcal{T} such that $y_i = f_i y$ is in the essential image of Φ for all i . The map Φ is called an *internal equivalence* if $\Phi(R)$ is full and faithful for all R , and if the sheaf-theoretic essential image of Φ is Y_0 itself.

For example, Φ is an internal equivalence in the trivial topology if and only if $\Phi(R)$ is full, faithful, and essentially surjective for all R , so that $\Phi(R)$ is an equivalence of groupoids for all R .

Our goal is then to prove the following theorem, which is Theorem B of the introduction.

Theorem 3.2. *Suppose $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is an internal equivalence of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$. Then $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\mathcal{T}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\mathcal{T}}$ is an equivalence of categories.*

As usual, our proof of this theorem will work in the graded case as well.

We point out that there should be a model structure on presheaves of groupoids extending the Joyal-Tierney model structure. The weak equivalences in this model structure would be the maps Φ which are sheaf-theoretically fully faithful and whose sheaf-theoretic essential image is all of Y_0 . Theorem 3.2 should then be a special case of the more general theorem that a weak equivalence of presheaves of groupoids induces an equivalence of their categories of sheaves. We have not considered this more general case, because $\mathrm{Spec} A$ is already a sheaf in the flat topology, and $\mathrm{Spec} A$ is our main object of interest.

We will prove this theorem by showing that Φ^* is full, faithful, and essentially surjective. The proof of each such step will be long, but divided into discrete steps very much like a diagram chase. In general, we are trying in each case to construct something for every point (R, y) of Y_0 . So first we do it for points (R, y) in the essential image of Φ . This generally involves choosing a point (R, x) of X_0 and a morphism $\alpha: \Phi x \rightarrow y$, so we generally have to prove that which choice one makes is immaterial. Then we show that every property we hope for in the construction is true on the essential image of Φ . Next we extend the definition to all points (R, y) in the sheaf-theoretic essential image of Φ by using a cover. Once again, this depends on the choice of cover, so we have to show the choice is immaterial. For this, it is enough to show that refining the cover makes no difference, since any two covers have a common refinement. Finally, we show that the properties we want are sheaf-theoretic in nature, so that since they hold already on the essential image of Φ , they also hold on the sheaf-theoretic essential image of Φ .

Proposition 3.3. *Suppose $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is an map of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$ whose sheaf-theoretic essential image is all of Y_0 . Then*

$$\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\mathcal{T}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\mathcal{T}}$$

is faithful.

Proof. Suppose $\tau: M \rightarrow N$ is a map of sheaves on (Y_0, Y_1) such that $\Phi^*\tau = 0$. This means that $\tau_{\Phi x} = 0$ for all points (R, x) of X_0 . We must show that $\tau_y = 0$ for all points (R, y) of Y_0 . We first show that $\tau_y = 0$ for all y in the essential image of Φ . Indeed, suppose α is a morphism from Φx to y . Then, since τ commutes with the structure map ψ , we get the commutative diagram below.

$$\begin{array}{ccc} M_{\Phi x} & \xrightarrow{\psi_{\alpha}^M} & M_y \\ \tau_{\Phi x} \downarrow & & \downarrow \tau_y \\ N_{\Phi x} & \xrightarrow{\psi_{\alpha}^N} & N_y \end{array}$$

It follows that $\tau_y = 0$.

Now suppose (R, y) is a general point of Y_0 . Since y is in the sheaf-theoretic essential image of Φ , we can choose a covering $\{R \xrightarrow{f_i} S_i\}$ such that $y_i = Y_0(f_i)(y)$ is in the essential image of Φ for all i . Thus $\tau_{y_i} = 0$ for all i . We then have a commutative diagram

$$\begin{array}{ccc} M_y & \longrightarrow & \prod M_{y_i} \\ \tau_y \downarrow & & \downarrow \prod \tau_{y_i} \\ N_y & \longrightarrow & \prod N_{y_i} \end{array}$$

The horizontal arrows are monomorphisms, since M and N are sheaves in \mathcal{T} , so $\tau_y = 0$ as well. \square

Note that we have actually shown, more generally, that if $\tau: M \rightarrow N$ is a morphism of sheaves over (Y_0, Y_1) such that $\Phi^*\tau = 0$, then τ restricted to the sheaf-theoretic essential image of Φ is also 0.

Proposition 3.4. *Suppose $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is an map of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$ whose sheaf-theoretic essential image is all of Y_0 and such that $\Phi(R)$ is full for all R . Then $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\mathcal{T}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\mathcal{T}}$ is full.*

Proof. Suppose we have a map $\tau: \Phi^*M \rightarrow \Phi^*N$. This means we have maps $\tau_x: M_{\Phi x} \rightarrow N_{\Phi x}$ for all points (R, x) of X_0 . We need to construct maps $\sigma_y: M_y \rightarrow N_y$ for all points (R, y) of Y_0 such that $\sigma_{\Phi x} = \tau_x$. Suppose first that y is in the essential image of Φ , so that there is a morphism α from Φx to y for some point (R, x) of X_0 . If σ were to exist, then we would have the commutative diagram below,

$$\begin{array}{ccc} M_{\Phi x} & \xrightarrow{\psi_{\alpha}^M} & M_y \\ \tau_x \downarrow & & \downarrow \sigma_y \\ N_{\Phi x} & \xrightarrow{\psi_{\alpha}^N} & N_y \end{array}$$

so we define $\sigma_y = \psi_{\alpha}^N \tau_x (\psi_{\alpha}^M)^{-1}$.

We claim that this definition of σ_y is independent of the choice of α . Indeed, suppose $\beta \in Y_1(R)$ is a morphism from $\Phi x'$ to y . Then $\beta^{-1}\alpha$ is a morphism from Φx to $\Phi x'$, and so, since Φ is full, there is a morphism $\gamma \in X_1(R)$ from x to x' such that $\Phi\gamma = \beta^{-1}\alpha$. Since τ is a map of sheaves, $\tau_{x'}\psi_{\Phi\gamma}^M = \psi_{\Phi\gamma}^N\tau_x$. On the other hand, by the cocycle condition we have $\psi_{\Phi\gamma} = (\psi_\beta)^{-1}\psi_\alpha$. Combining these two equations gives

$$\psi_\alpha^N\tau_x(\psi_\alpha^M)^{-1} = \psi_\beta^N\tau_{x'}(\psi_\beta^M)^{-1},$$

so σ_y is independent of the choice of α . In particular, if $y = \Phi x$, we can take α to be the identity map of Φx . The cocycle condition implies that ψ_α^M and ψ_α^N are identity maps, and so $\sigma_{\Phi x} = \tau_x$.

We now show that σ commutes with the structure maps of M and N on the essential image of Φ . Suppose that $(R, y) \xrightarrow{f} (S, y')$ is a map of points of Y_0 , and that y is in the essential image of Φ . Choose a morphism α from Φx to y for some point (R, x) of X_0 . Let $\alpha' = Y_1(f)(\alpha)$, so that α' is a morphism from $\Phi x'$ to y' , where $x' = X_0(f)(x)$. Since τ is a map of sheaves, we get the commutative square below.

$$\begin{array}{ccc} M_{\Phi x} & \xrightarrow{\tau_x} & N_{\Phi x} \\ \theta^M(f, \Phi x) \downarrow & & \downarrow \theta^N(f, \Phi x) \\ M_{\Phi x'} & \xrightarrow{\tau_{x'}} & N_{\Phi x'} \end{array}$$

We would like to know that the square below is commutative.

$$\begin{array}{ccc} M_y & \xrightarrow{\sigma_y} & N_y \\ \theta^M(f, y) \downarrow & & \downarrow \theta^N(f, y) \\ M_{y'} & \xrightarrow{\sigma_{hy}} & N_{y'}. \end{array}$$

We claim that is an isomorphism from the top square to the bottom square, and so the bottom square must be commutative. Indeed, in the upper left corner this isomorphism is ψ_α^M , in the upper right corner it is ψ_α^N , in the lower left corner it is $\psi_{\alpha'}^M$, and in the lower right corner it is $\psi_{\alpha'}^N$. All the required diagrams commute to make this a map of squares. This uses the fact that ψ^M and ψ^N are maps of sheaves and the well-definedness of σ .

We now check that σ commutes with ψ , on the essential image of Φ . Suppose we have a morphism $\beta: y \rightarrow y'$ in $(Y_0(R), Y_1(R))$, and that y is in the essential image of Φ . Let α be a morphism from Φx to y for some point (R, x) of X_0 . Consider the following diagram.

$$\begin{array}{ccccc} M_{\Phi x} & \xrightarrow{\psi_\alpha^M} & M_y & \xrightarrow{\psi_\beta^M} & M_{y'} \\ \tau_x \downarrow & & \sigma_y \downarrow & & \downarrow \sigma_{y'} \\ N_{\Phi x} & \xrightarrow{\psi_\alpha^N} & N_y & \xrightarrow{\psi_\beta^N} & N_{y'} \end{array}$$

By definition of σ , the left-hand square is commutative. The cocycle condition implies that $\psi_\beta \circ \psi_\alpha = \psi_{\beta\alpha}$, so the definition of σ also implies that the outside

square commutes. Since the horizontal maps are isomorphisms, the right-hand square must also be commutative.

We now extend the definition of σ to an arbitrary point (R, y) of Y_0 . The sheaf-theoretic essential image of Φ is all of Y_0 , we can choose a cover $\{R \xrightarrow{f_i} S_i\}$ of R in the topology \mathcal{T} such that $y_i = Y_0(f_i)(y)$ is in the essential image of Φ for all i . Let y_{jk} denote the image of y in $Y_0(S_j \otimes_R S_k)$. We then have a commutative diagram

$$\begin{array}{ccccc} M_y & \longrightarrow & \prod M_{y_i} & \longrightarrow & \prod M_{y_{jk}} \\ & & \downarrow \Pi \sigma_{y_i} & & \downarrow \Pi \sigma_{y_{jk}} \\ N_y & \longrightarrow & \prod N_{y_i} & \longrightarrow & \prod N_{y_{jk}} \end{array}$$

where the right-hand horizontal maps are the difference of the two restriction maps. Thus each row expresses its left-hand entry as a kernel. The diagram commutes since σ is a map of sheaves on the essential image of Φ . Thus, there is a unique map $\sigma_y: M_y \rightarrow N_y$ making the diagram commute.

We now check that σ_y is independent of the choice of cover. It suffices to show that σ_y is unchanged if we replace the cover $\{R \rightarrow S_i\}$ by a refinement $\{R \rightarrow T_j\}$, since any two covers have a common refinement. If we denote the map coming from the refinement by σ'_y , then we would have to have $\sigma'_{y_i} = \sigma_{y_i}$, since some of the T_j form a cover of S_i and σ is a map of sheaves on the essential image of Φ . Then the sheaf condition forces $\sigma'_y = \sigma_y$ as well. In particular, if y is already in the essential image of Φ , then we can take the identity cover to find that the new definition of σ is an extension of our old definition.

We now show that σ is a map of sheaves over Y_0 . Suppose we have a map $(R, y) \xrightarrow{f} (S, y')$ of points of Y_0 . Choose a cover $\{R \xrightarrow{g_i} T_i\}$ of R such that $y_i = Y_0(g_i)(y)$ is in the essential image of Φ for all i . Then there is an induced cover $\{S \xrightarrow{h_i} U_i = S \otimes_R T_i\}$ of S . The map f induces corresponding maps $f_i: (T_i, y_i) \rightarrow (U_i, y'_i)$, where $y'_i = Y_0(h_i)(y')$. Since σ is a map of sheaves on the essential image of Φ , we have the commutative diagram below.

$$\begin{array}{ccc} M_{y_i} & \xrightarrow{\sigma_{y_i}} & N_{y_i} \\ \downarrow & & \downarrow \\ M_{y'_i} & \xrightarrow{\sigma_{y'_i}} & N_{y'_i} \end{array}$$

The sheaf condition and the definition of σ then show that the diagram

$$\begin{array}{ccc} M_y & \xrightarrow{\sigma_y} & N_y \\ \downarrow & & \downarrow \\ M_{y'} & \xrightarrow{\sigma_{y'}} & N_{y'} \end{array}$$

is commutative, and so σ is a map of sheaves over Y_0 .

The proof that σ commutes with ψ , and so is a map of sheaves over (Y_0, Y_1) , is similar. \square

Finally, we show that Φ^* is essentially surjective.

Proposition 3.5. *Suppose $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is an internal equivalence of presheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$. Then $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\mathcal{T}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\mathcal{T}}$ is essentially surjective.*

Proof. Suppose that N is a sheaf over (X_0, X_1) . We must construct a sheaf M over (Y_0, Y_1) and an isomorphism $\Phi^*M \rightarrow N$ of sheaves. We first construct M_y for y in the essential image of Φ , and show that it has the desired properties there. For every point (R, y) in the essential image of Φ , choose a point $(R, x(y))$ of X_0 and a morphism $\alpha(y)$ from $x(y)$ to y . Note that this only requires choosing over a set, since \mathbf{Aff} is a small category. Define $M_y = N_{x(y)}$.

We now construct the restriction of the structure map θ^M to the essential image of Φ . Suppose that we have a map $(R, y) \xrightarrow{f} (S, y')$ between points of Y_0 , where (R, y) is in the essential image of Φ . Let $\alpha' = Y_1(f)(\alpha(y))$, so that α' is a morphism from $\Phi x'$ to y' , where $x' = X_0(f)(x(y))$. Then $\alpha(y')^{-1}\alpha'$ is a morphism from $\Phi x'$ to $\Phi x(y')$. Since Φ is full and faithful, there is a unique morphism γ of $X_1(S)$ from x' to $x(y')$ such that $\Phi\gamma = \alpha(y')^{-1}\alpha'$. We then define $\theta^M(f, y): M_y \rightarrow M_{y'}$ to be the composite

$$M_y = N_{x(y)} \xrightarrow{\theta^N(f, x(y))} N_{x'} \xrightarrow{\psi_{\gamma}^N} N_{x(y')} = M_{y'}.$$

We must check the functoriality conditions for θ^M (restricted to the essential image of Φ). First of all, if f is the identity map, then $\Phi\gamma$ will be the identity morphism of y . Since Φ is faithful, it follows that γ is the identity morphism of $x(y)$. The cocycle condition forces ψ_{γ}^N to be the identity map, and so $\theta^M(1, y)$ is the identity as required. If $g: (S, y') \rightarrow (T, y'')$ is another map of points of Y_0 , a diagram chase involving the cocycle condition for ψ^N and the fact that ψ^N is a map of sheaves shows that $\theta^M(gf, y)$ is the composition $\theta^M(g, y')\theta^M(f, y)$.

We now show that M is a sheaf on the essential image of Φ . Indeed, suppose (R, y) is a point in the essential image of Φ , and $\{R \rightarrow S_i\}$ is a cover of R in \mathcal{T} . We must check that

$$M_y \rightarrow \prod M_{y_i} \rightrightarrows \prod M_{y_{jk}}$$

is an equalizer diagram. We have an equalizer diagram

$$M_y = N_{x(y)} \rightarrow \prod N_{x(y)_i} \rightrightarrows \prod N_{x(y)_{jk}}$$

since N is a sheaf. We construct an isomorphism from the bottom diagram to the top, from which it follows that the top is also an equalizer diagram. The morphism $\alpha(y): \Phi x(y) \rightarrow y$ induces a morphism $\alpha(y)_i: \Phi x(y)_i \rightarrow y_i$. We also have the morphism $\alpha(y_i): \Phi x(y_i) \rightarrow y_i$. The composition $(\alpha(y_i))^{-1} \circ \alpha(y)_i = \Phi\gamma$ for a unique $\gamma: x(y)_i \rightarrow x(y_i)$, since Φ is full and faithful. Then $\psi_{\gamma}: N_{x(y)_i} \rightarrow N_{x(y_i)} = M_{y_i}$ defines the desired isomorphism $\prod N_{x(y)_i} \rightarrow \prod M_{y_i}$. One constructs the isomorphism $\prod N_{x(y)_{jk}} \rightarrow \prod N_{x(y_{jk})} = \prod M_{y_{jk}}$ in the same manner, using the morphisms $\alpha(y)_{jk}: \Phi x(y)_{jk} \rightarrow y_{jk}$ and $\alpha(y_{jk})$. The proof that the diagram below

$$\begin{array}{ccc} N_{x(y_i)} = M_{y_i} & \longrightarrow & N_{x(y_{ij})} = M_{y_{ij}} \\ \uparrow & & \uparrow \\ N_{x(y)_i} & \longrightarrow & N_{x(y)_{ij}} \end{array}$$

is commutative is a computation using the fact that ψ^N is a map of sheaves, the cocycle condition, and the fact that Φ is faithful.

We now construct the restriction of the map ψ^M to the essential image of Φ . Suppose β is a morphism from y to y' , where y is in the essential image of Φ . Then $\alpha(y')^{-1}\beta\alpha(y)$ is a morphism from $\Phi x(y)$ to $\Phi x(y')$. Since Φ is full and faithful, there is a unique morphism γ from $x(y)$ to $x(y')$ such that $\Phi\gamma = \alpha(y')^{-1}\beta\alpha(y)$. Hence we can define $\psi_\beta^M = \psi_\gamma^N$. We leave to the reader the diagram chase showing that ψ is a map of sheaves.

We now construct the desired isomorphism of sheaves $\tau: \Phi^*M \rightarrow N$. (Since Φ^*M is determined by the restriction of M to the image of Φ , we can do this even though we have not completed the definition of M). Suppose (R, x) is a point of X_0 . Then $\alpha(\Phi x)$ is a morphism from $\Phi(x(\Phi x))$ to Φx . Since Φ is full and faithful, there is a unique morphism β from $x(\Phi x)$ to x such that $\Phi\beta = \alpha(\Phi x)$. We define

$$\tau_x = \psi_\beta^N: M_{\Phi x} = N_{x(\Phi x)} \rightarrow N_x.$$

Obviously τ_x is an isomorphism, but we must check it is compatible with the structure maps. We leave these checks to the reader; both are diagram chases.

We have now defined a sheaf M on the essential image of Φ , and to complete the proof we need only extend it to a sheaf on all of (Y_0, Y_1) . For each point (R, y) of Y_0 , choose a cover $C(y) = \{R \xrightarrow{f_i} S_i\}$ such that $y_i = Y_0(f_i)(y)$ is in the essential image of Φ for all i , making sure to choose the identity cover when y is already in the essential image of Φ . Once again, we can do this since \mathbf{Aff} is a small category. We then define M_y as we must if we are going to get a sheaf, as the equalizer of the two maps of R -modules

$$\prod_i M_{y_i} \rightrightarrows \prod_{jk} M_{y_{jk}}.$$

This definition of M_y will of course depend on the choice of cover $C(y)$. Suppose $D = \{R \rightarrow T_m\}$ is some other cover such that y_m is in the essential image of Φ for all m . We claim that there is a canonical equalizer diagram

$$M_y \rightarrow \prod M_{y_m} \rightrightarrows \prod M_{y_{np}}.$$

To see this, let M_y^D denote the pullback of the two arrows

$$\prod_m M_{y_m} \rightrightarrows \prod_{np} M_{y_{np}}.$$

We claim that there is a canonical isomorphism $M_y^D \rightarrow M_y$. It suffices to check this when D is a refinement of $C(y)$, since any two covers have a common refinement. In this case, there is a diagram

$$M_y \rightarrow \prod_m M_{y_m} \rightrightarrows \prod_{np} M_{y_{np}},$$

where the first map is induced by first mapping to M_{y_i} , and then using the structure maps of M restricted to the essential image of Φ to map further to M_{y_m} . It suffices to prove that this diagram is an equalizer. It is easy to check that M_y maps into the equalizer. If $t \in M_y$ maps to 0 in each M_{y_m} , then, using the fact that M restricted to the essential image of Φ is a sheaf, we find that t maps to 0 in each M_{y_i} . By definition of M_y , then, $t = 0$. Similarly, suppose $(t_m) \in \prod M_{y_m}$ is in the equalizer. Again using the fact that M restricted to the essential image of Φ is a sheaf, we construct an element $(t_i) \in \prod M_{y_i}$. The images of t_i and t_j in $M_{y_{ij}}$ coincide, since they coincide after restriction to the induced cover. Thus we get an element $t \in M_y$

restricting to the t_i . It follows that t restricts to the t_m as well, and so M_y is the desired equalizer.

Now we can construct the structure maps of M . Suppose $(R, y) \rightarrow (S, z)$ is a map of points of Y_0 . The cover $C(y) = \{R \rightarrow S_i\}$ of R induces a cover $D = \{S \rightarrow S \otimes_R S_i\}$ of S , and the restriction z_i of z is in the essential image of Φ for all i , since y_i is so. Thus we get a map from

$$\prod M_{y_i} \rightrightarrows \prod M_{y_{jk}}$$

to

$$\prod M_{z_i} \rightrightarrows \prod M_{z_{jk}},$$

and so an induced map $M_y \rightarrow M_z^D$ on the equalizers. After composing this with the canonical isomorphism $M_z^D \rightarrow M_z$, we get the desired structure map $\theta: M_y \rightarrow M_z$. Since we chose the identity cover when y was already in the essential image of Φ , this extends the definition we have already given in that case. We leave it to the reader to check the functoriality of θ .

We now show that M is a sheaf. Suppose (R, y) is a point of Y_0 and $\{(R, y) \rightarrow (T_m, y_m)\}$ is a cover of R . Let $C(y) = \{(R, y) \rightarrow (S_i, y_i)\}$ be the given cover of R , so that each y_i is in the essential image of Φ . Then $\{S_i \rightarrow T_m \otimes_R S_i\}$ is a cover of S_i , and each y_{mi} is the essential image of Φ since each y_i is. Similarly, $\{T_m \rightarrow T_m \otimes_R S_i\}$ is a cover of T_m . Thus we get the commutative diagram below.

$$\begin{array}{ccccc} M_y & \longrightarrow & \prod_m M_{y_m} & \longrightarrow & \prod_{np} M_{y_{np}} \\ \downarrow & & \downarrow & & \downarrow \\ \prod_i M_{y_i} & \longrightarrow & \prod_{mi} M_{y_{mi}} & \longrightarrow & \prod_{npi} M_{y_{npi}} \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{jk} M_{y_{jk}} & \longrightarrow & \prod_{mjk} M_{y_{mjk}} & \longrightarrow & \prod_{npj k} M_{y_{npj k}} \end{array}$$

The subscripts m , n , and p all refer to the T_m , and the subscripts i , j and k all refer to the S_i . So, for example, y_{npi} is the image of y in $Y_0(T_n \otimes_R T_p \otimes_R S_i)$. The right-hand horizontal arrows are all the differences of the two restriction maps. This means that the second and third rows express their left-hand entries as kernels, since M restricted to the essential image of Φ is a sheaf. Similarly, the bottom vertical arrows are also differences of the two restriction maps. It follows that each column expresses its top entry as a kernel, since the definition of M does not depend on which cover we choose, up to isomorphism. A diagram chase then shows that the top row expresses M_y as a kernel, which means that M is a sheaf.

We now construct the isomorphism $\psi: \text{dom}^* M \rightarrow \text{codom}^* M$. Suppose $\alpha: y \rightarrow z$ is a morphism in $Y_1(R)$. Let $\{R \rightarrow S_i\}$ be the given cover of (R, y) , so that each y_i is in the essential image of Φ . It follows that z_i is also in the essential image of Φ for all i . Let $\alpha_i: y_i \rightarrow z_i$ denote the image of α in $Y_1(S_i)$, and similarly let α_{jk} denote the image of α in $Y_1(S_j \otimes_R S_k)$. Then we have a commutative diagram

$$\begin{array}{ccccc} M_y & \longrightarrow & \prod M_{y_i} & \longrightarrow & \prod M_{y_{jk}} \\ & & \prod \psi_{\alpha_i} \downarrow & & \downarrow \prod \psi_{\alpha_{jk}} \\ M_z & \longrightarrow & \prod M_{z_i} & \longrightarrow & \prod M_{z_{jk}} \end{array}$$

Here the right-hand horizontal arrows are differences of restriction maps, as usual. The top row is an equalizer by definition, and we have proved that the bottom row is also an equalizer diagram. Hence there is a unique map $\psi_\alpha: M_y \rightarrow M_z$, necessarily an isomorphism, making the diagram commute. The facts that ψ satisfies the cocycle condition and is a map of sheaves are the usual sheaf-theoretic diagram chases, and we leave them to the reader. \square

4. QUASI-COHERENT SHEAVES

The object of this section is to prove Theorem C, showing that if $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is an internal equivalence of presheaves of groupoids in the flat topology, then $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\text{qc}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\text{qc}}$ is an equivalence of categories of quasi-coherent sheaves. This theorem can be viewed as a manifestation of faithfully flat descent; we have seen already that $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^T \rightarrow \mathbf{Sh}_{(X_0, X_1)}^T$ is an equivalence of categories, and we use faithfully flat descent to conclude that quasi-coherent sheaves are a full subcategory of sheaves in the flat topology.

Recall that a cover of R in the flat, or fpqc, topology is a *finite* collection of maps $\{R \rightarrow S_i\}$ such that each S_i is flat over R , and the product $\prod S_i$ is faithfully flat over R . This also defines the flat topology on \mathbf{Aff}_* .

We use faithfully flat descent in the form of the following well-known lemma.

Lemma 4.1. *Suppose $\{R \rightarrow S_i\}$ is a cover of R in the flat topology on \mathbf{Aff} , and M is an R -module. Then the diagram*

$$M \rightarrow \prod_i S_i \otimes_R M \rightrightarrows \prod_{jk} S_j \otimes_R S_k \otimes_R M$$

is an equalizer in the category of R -modules.

Of course, the two maps in the equalizer take $s \otimes m \in S_i \otimes M$ to $(1 \otimes s_i \otimes m) \in \prod_{ji} S_j \otimes_R S_i \otimes_R M$ and to $s_i \otimes 1 \otimes m \in \prod_{ik} S_i \otimes_R S_k \otimes_R M$.

As usual, this lemma also works in the graded case, with the same proof.

Proof. Let $S = \prod_i S_i$. Since the product is finite, it suffices to show that

$$M \rightarrow S \otimes_R M \rightrightarrows S \otimes_R S \otimes_R M$$

is an equalizer for all R -modules M . Since S is faithfully flat, it suffices to show that

$$S \otimes_R M \rightarrow S \otimes_R S \otimes_R M \rightrightarrows S \otimes_R S \otimes_R S \otimes_R M$$

is an equalizer for all M . But, before tensoring with M , this sequence is just the beginning of the bar resolution of S as an R -algebra; since the bar resolution is contractible, this diagram remains an equalizer after tensoring with M . \square

Lemma 4.1 leads immediately to the following proposition, which is also true in the graded case.

Proposition 4.2. *Suppose M is a quasi-coherent sheaf over a presheaf of groupoids (X_0, X_1) on \mathbf{Aff} . Then M is a sheaf in the flat topology.*

Proof. Suppose (R, y) is a point of X_0 , and $\{(R, y) \rightarrow (S_i, y_i)\}$ is a cover in the flat topology. We must show that the diagram

$$E_y = (M_y \rightarrow \prod M_{y_i} \rightrightarrows \prod M_{y_{jk}})$$

is an equalizer diagram. But, since M is quasi-coherent, E_y is isomorphic to the diagram

$$M_y \rightarrow \prod S_i \otimes_R M_y \rightrightarrows \prod S_j \otimes_R S_k \otimes_R M_y,$$

which is an equalizer diagram by Lemma 4.1. \square

We will also need a lemma about purity of equalizer diagrams.

Definition 4.3. Suppose E is an equalizer diagram of the form

$$A \rightarrow B \rightrightarrows C$$

in the category of R -modules for some commutative ring R . We say that E is *pure* if $S \otimes_R E$ is still an equalizer diagram for all commutative R -algebras S .

One can also define purity using arbitrary R -modules S . We prefer this definition because it is the concept we need, but in fact the two definitions are equivalent. Either definition also works in the graded case with the obvious changes.

Lemma 4.4. *Suppose E is an equalizer diagram of R -modules for some commutative ring R . Suppose $\{S_i\}$ is a set of flat commutative R -algebra such that $S_i \otimes_R E$ is pure for all i and $S = \bigoplus_i S_i$ is faithfully flat over R . Then E is pure.*

Proof. Suppose T is an arbitrary R -algebra. Then $(T \otimes_R S_i) \otimes_{S_i} (S_i \otimes_R E)$ is an equalizer diagram since $S_i \otimes_R E$ is pure, but

$$(T \otimes_R S_i) \otimes_{S_i} (S_i \otimes_R E) \cong (T \otimes_R S_i) \otimes_T (T \otimes_R E).$$

Thus $(T \otimes_R S) \otimes_T (T \otimes_R E)$ is also an equalizer diagram, being a direct sum of equalizer diagrams. Since $T \otimes_R S$ is faithfully flat over T , it follows that $T \otimes_R E$ is an equalizer diagram. \square

We can now prove that quasi-coherent sheaves are homotopy invariant in the flat topology. The following theorem is Theorem C of the introduction.

Theorem 4.5. *Suppose $\Phi: (X_0, X_1) \rightarrow (Y_0, Y_1)$ is an internal equivalence of pre-sheaves of groupoids on $\mathbf{Aff}_{\mathcal{T}}$, where \mathcal{T} is the flat topology. Then $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\text{qc}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\text{qc}}$ is an equivalence of categories.*

This theorem is also true in the graded case, with the same proof.

Proof. Since $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\mathcal{T}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\mathcal{T}}$ is an equivalence of categories, and quasi-coherent sheaves are a full subcategory of sheaves in the flat topology by Proposition 4.2, we find immediately that $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\text{qc}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\text{qc}}$ is full and faithful. It remains to show that it is essentially surjective.

Suppose N is a quasi-coherent sheaf over (X_0, X_1) . Because $\Phi^*: \mathbf{Sh}_{(Y_0, Y_1)}^{\mathcal{T}} \rightarrow \mathbf{Sh}_{(X_0, X_1)}^{\mathcal{T}}$ is an equivalence of categories, there is a sheaf M in the flat topology, over (Y_0, Y_1) , such that $\Phi^* M \cong N$. We will show that M is in fact quasi-coherent, so that Φ^* is essentially surjective on quasi-coherent sheaves. To do so, we must show that, if $(R, y) \xrightarrow{f} (S, y')$ is a map of points of Y_0 , then the adjoint $S \otimes_R M_y \xrightarrow{\rho^M(f)} M_{y'}$ of the structure map of M is an isomorphism.

First suppose that y is in the essential image of Φ . Then there is an $x \in X_0(R)$ and a map $\alpha: \Phi x \rightarrow y$. Let $x' = f(x) \in X_0(S)$, so that $f(\alpha) = X_1(f)(\alpha): \Phi x' \rightarrow$

z. Then we have the commutative diagram below.

$$\begin{array}{ccc}
S \otimes_R N_x & \xrightarrow{\rho^N(f)} & N_{x'} \\
\cong \downarrow & & \downarrow \cong \\
S \otimes_R M_{\Phi x} & \xrightarrow{\rho^M(f)} & M_{\Phi x'} \\
1 \otimes \psi_\alpha \downarrow & & \downarrow \psi_{f\alpha} \\
S \otimes_R M_y & \xrightarrow{\rho^M(f)} & M_{y'}
\end{array}$$

The top square of this diagram commutes because $\Phi^*M \cong N$ as sheaves, and the bottom square commutes because ψ is a map of sheaves. The vertical maps are isomorphisms, and the top horizontal map is an isomorphism since N is quasi-coherent. Hence the bottom horizontal map is an isomorphism as well.

In fact, if y is in the essential image of Φ and $\{R \rightarrow S_i\}$ is a cover of R in the flat topology, we claim that the equalizer diagram

$$(4.6) \quad E = E_y = (M_y \rightarrow \prod M_{y_i} \rightrightarrows \prod M_{y_{jk}})$$

is pure. Indeed, suppose S is an R -algebra, so we have $f: (R, y) \rightarrow (S, y')$. Then $\{S \rightarrow S \otimes_R S_i\}$ is a cover of S in the flat topology. It follows from what we have just done (and the fact that covers in the flat topology are finite), that the diagram $S \otimes_R E_y$ is isomorphic to $E_{y'}$, and so is still an equalizer diagram.

Now suppose y is an arbitrary point of Y_0 . Since the sheaf-theoretic essential image of Φ is all of Y_0 , we can choose a cover $\{R \rightarrow S_i\}$ such that each y_i is in the essential image of Φ . There is an induced cover $\{S \rightarrow S \otimes_R S_i\}$ of S , and maps $f_i: (S_i, y_i) \rightarrow (S \otimes_R S_i, y'_i)$, so each y'_i is also in the essential image of Φ . We then get the commutative diagram below, which is a map from the diagram $S \otimes_R E_y$ to E_z .

$$\begin{array}{ccccc}
S \otimes_R M_y & \longrightarrow & \prod S \otimes_R M_{y_i} & \xrightarrow{1 \otimes d} & \prod S \otimes_R M_{y_{jk}} \\
\rho_f \downarrow & & \prod \rho(f_i) \downarrow & & \downarrow \prod \rho(f_{jk}) \\
M_z & \longrightarrow & \prod M_{z_i} & \xrightarrow{d} & \prod M_{z_{jk}}.
\end{array}$$

Here the map d is the difference between the two restriction maps, so the bottom row expresses M_z as a kernel. We have already seen that the maps $\rho(f_i)$ and $\rho(f_{jk})$ are isomorphisms, so if we knew that $S \otimes_R E_y$ were an equalizer diagram, we would be able to conclude that $\rho(f)$ is an isomorphism, and therefore that M is quasi-coherent.

In particular, if S is flat over R , we conclude that the diagram $S \otimes_R E_y$ is isomorphic to the equalizer diagram $E_{y'}$. In case y' is in the essential image of Φ , we have proved that $E_{y'}$ is pure. In particular, $S_i \otimes_R E$ is a pure equalizer diagram for all i . Since $\prod S_i$ is faithfully flat over R , it follows from Lemma 4.4 that the equalizer diagram E is pure. Thus, for any S , $S \otimes_R E$ is an equalizer diagram, and so M is quasi-coherent. \square

5. HOPF ALGEBROIDS

In this section, we prove Theorem D of the introduction, characterizing those maps of Hopf algebroids which induce internal equivalences in the flat topology of the corresponding presheaves of groupoids.

Suppose $f = (f_0, f_1): (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of Hopf algebroids. See [Rav86, Definition A1.1.7] for an explicit definition of this, though of course f is equivalent to a map $\Phi = f^*: (\text{Spec } B, \text{Spec } \Sigma) \rightarrow (\text{Spec } A, \text{Spec } \Gamma)$ of sheaves of groupoids on \mathbf{Aff} . A map of Hopf algebroids induces a map

$$B \otimes_A \Gamma \otimes_A B \xrightarrow{\eta_L \otimes f_1 \otimes \eta_R} \Sigma_{\eta_L f_0} \otimes_A f_1 \eta_L \Sigma_{f_1 \eta_R} \otimes_A \eta_R f_0 \Sigma \xrightarrow{\mu} \Sigma,$$

where μ denotes multiplication. Note that μ makes sense since $f_1 \eta_L = \eta_L f_0$ and $f_1 \eta_R = \eta_R f_0$. By abuse of notation, we denote this map simply by $\eta_L \otimes f_1 \otimes \eta_R$.

Our goal is to characterize those f for which f^* is a weak equivalence. We begin by determining when f^* is faithful.

Proposition 5.1. *Suppose $f = (f_0, f_1): (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of Hopf algebroids. Then $f^*: (\text{Spec } B, \text{Spec } \Sigma) \rightarrow (\text{Spec } A, \text{Spec } \Gamma)$ is faithful if and only if $\eta_L \otimes f_1 \otimes \eta_R: B \otimes_A \Gamma \otimes_A B \rightarrow \Sigma$ is an epimorphism in **Rings**.*

Recall that an epimorphism in **Rings** need not be surjective; the map from the integers to the rational numbers is a ring epimorphism. Also note that the obvious generalization of this proposition holds for graded Hopf algebroids.

Proof. Given $\alpha, \beta: \Sigma \rightarrow R$,

$$\alpha \circ (\eta_L \otimes f_1 \otimes \eta_R) = \beta \circ (\eta_L \otimes f_1 \otimes \eta_R)$$

if and only if α and β have the same domain and codomain when thought of as morphisms of $(\text{Spec } B, \text{Spec } \Sigma)(R)$ and $f^* \alpha = f^* \beta$. The proposition follows. \square

We now determine when f^* is full.

Proposition 5.2. *Suppose $f = (f_0, f_1): (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of Hopf algebroids. Then $f^*: (\text{Spec } B, \text{Spec } \Sigma) \rightarrow (\text{Spec } A, \text{Spec } \Gamma)$ is full if and only if $\eta_L \otimes f_1 \otimes \eta_R: B \otimes_A \Gamma \otimes_A B \rightarrow \Sigma$ is a split monomorphism of rings.*

Once again, the obvious generalization of this proposition is true in the graded case.

Proof. The map f^* is full if and only if every morphism

$$\beta: f^* x \rightarrow f^* y \in (\text{Spec } A, \text{Spec } \Gamma)(R)$$

is equal to $f^* \alpha$ for some morphism $\alpha: x \rightarrow y$ of $(\text{Spec } B, \text{Spec } \Sigma)(R)$. Said another way, f^* is full if and only if every ring homomorphism

$$x \otimes \beta \otimes y: B \otimes_A \Gamma \otimes_A B \rightarrow R$$

can be extended through $\eta_L \otimes f_1 \otimes \eta_R$ to a ring homomorphism $\Sigma \rightarrow R$. This is equivalent to $\eta_L \otimes f_1 \otimes \eta_R$ being a split monomorphism. \square

Corollary 5.3. *Suppose $f = (f_0, f_1): (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of Hopf algebroids. Then $f^*: (\text{Spec } B, \text{Spec } \Sigma) \rightarrow (\text{Spec } A, \text{Spec } \Gamma)$ is fully faithful if and only if $\eta_L \otimes f_1 \otimes \eta_R: B \otimes_A \Gamma \otimes_A B \rightarrow \Sigma$ is an isomorphism.*

Proof. Any map $g: R \rightarrow S$ of rings that is both a split monomorphism and a ring epimorphism is an isomorphism. Indeed, $\mathbf{Rings}(g, T): \mathbf{Rings}(S, T) \rightarrow \mathbf{Rings}(R, T)$ is monic since g is a ring epimorphism and epic since g is a split monomorphism, so is an isomorphism for all T . \square

Finally, we need to determine the sheaf-theoretic essential image of f^* is all of $\text{Spec } A$. For this we need the map $f_0 \otimes \eta_R: A \rightarrow B \otimes_A \Gamma$ defined as the composite

$$A \cong A \otimes_A A \xrightarrow{f_0 \otimes \eta_R} B \otimes_A \Gamma.$$

Proposition 5.4. *Suppose $f = (f_0, f_1): (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of Hopf algebroids. Then the sheaf-theoretic essential image of*

$$f^*: (\text{Spec } B, \text{Spec } \Sigma) \rightarrow (\text{Spec } A, \text{Spec } \Gamma)$$

is all of $\text{Spec } A$ if and only if there is a ring map $g: B \otimes_A \Gamma \rightarrow C$ such that $g(f_0 \otimes \eta_R)$ exhibits C as a faithfully flat extension of A .

This proposition is also true in the graded case, with the same proof.

Proof. We first determine when $y: A \rightarrow R$ is in the essential image of f^* . For this to happen we need an object $x: B \rightarrow R$ and a morphism $\alpha: \Gamma \rightarrow R$ from f^*x to y . A morphism α from f^*x to anywhere is equivalent to the composite

$$B \otimes_A \Gamma \xrightarrow{x \otimes \alpha} R_{x f_0} \otimes_A \alpha \eta_L R \xrightarrow{\mu} R,$$

which we also denote, by abuse of notation, by $x \otimes \alpha$. The codomain of α is the composite $(x \otimes \alpha)(f_0 \otimes \eta_R): A \rightarrow R$. Altogether then, y is in the essential image of f^* if and only if there is a map $h: B \otimes_A \Gamma \rightarrow C$ such that $h(f_0 \otimes \eta_R) = y$.

Now, suppose the sheaf-theoretic essential image of f^* is all of $\text{Spec } A$. Then there must be a cover $\{A \xrightarrow{h_i} S_i\}$ such that the image of the identity map of A , namely h_i , is in the essential image of f^* for all i . By the preceding paragraph, this is true if and only if there exist maps $g_i: B \otimes_A \Gamma \rightarrow S_i$ such that $g_i(f_0 \otimes \eta_R) = h_i$. Let C be the product of the S_i and let $g: B \otimes_A \Gamma \rightarrow C$ be the product of the g_i . Then $g(f_0 \otimes \eta_R)$ is the product of the h_i , which displays C as a faithfully flat extension of A since $\{A \xrightarrow{h_i} S_i\}$ is a cover of A .

Conversely, suppose there is a ring map $g: B \otimes_A \Gamma \rightarrow C$ such that $h = g(f_0 \otimes \eta_R)$ exhibits C as a faithfully flat extension of A . Suppose $y: A \rightarrow R$ is an arbitrary point of $(\text{Spec } A, \text{Spec } \Gamma)(R)$. Then

$$R \cong A \otimes_A R \xrightarrow{h \otimes 1} C \otimes_A R$$

is a cover of R . One can easily check that the image of y in $(\text{Spec } A, \text{Spec } \Gamma)(C \otimes_A R)$ is the composite

$$A \xrightarrow{h} C \cong C \otimes_A A \xrightarrow{1 \otimes y} C \otimes_A R.$$

Since $h = g(f_0 \otimes \eta_R)$, the image of y is in the essential image of f^* , and so y is in the sheaf-theoretic essential image of f^* . \square

Note that the proof of Proposition 5.4 can be easily modified to prove the known result that f^* is essentially surjective if and only if $f_0 \otimes \eta_R: A \rightarrow B \otimes_A \Gamma$ is a split monomorphism.

Altogether then, we have the following theorem, which is Theorem D of the introduction.

Theorem 5.5. *Suppose $f = (f_0, f_1): (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of Hopf algebroids. Then $f^*: (\text{Spec } B, \text{Spec } \Sigma) \rightarrow (\text{Spec } A, \text{Spec } \Gamma)$ is an internal equivalence in the flat topology if and only if*

$$\eta_L \otimes f_1 \otimes \eta_R: B \otimes_A \Gamma \otimes_A B \rightarrow \Sigma$$

is an isomorphism and there is a ring map $g: B \otimes_A \Gamma \rightarrow C$ such that $g(f_0 \otimes \eta_R)$ exhibits C as a faithfully flat extension of A .

This characterization of internal equivalences shows in particular that Σ is determined by (A, Γ) and f_0 . In fact, if (A, Γ) is any Hopf algebroid, and $f: A \rightarrow B$ is a ring homomorphism, there is a unique (up to isomorphism) Hopf algebroid (B, Γ_f) and map of Hopf algebroids (f, f_1) such that the map $\eta_L \otimes f_1 \otimes \eta_R$ is an isomorphism. To show existence, we take $\Gamma_f = B \otimes_A \Gamma \otimes_A B$ and define the structure maps as follows:

$$\begin{aligned} \eta_L: B &\cong B \otimes_A A \otimes_A A \xrightarrow{1 \otimes \eta_L \otimes f} B \otimes_A \Gamma \otimes_A B; \\ \eta_R: B &\cong A \otimes_A A \otimes_A B \xrightarrow{f \otimes \eta_R \otimes 1} B \otimes_A \Gamma \otimes_A B; \\ \epsilon: B \otimes_A \Gamma \otimes_A B &\xrightarrow{1 \otimes \epsilon \otimes 1} B \otimes_A A \otimes_A B \cong B \otimes_A B \xrightarrow{\mu} B; \\ c: B \otimes_A \Gamma \otimes_A B &\xrightarrow{1 \otimes c \otimes 1} B \otimes_A \eta_R \Gamma_{\eta_L} \otimes_A B \xrightarrow{\tau} B \otimes_A \eta_L \Gamma_{\eta_R} \otimes_A B; \\ \Delta: B \otimes_A \Gamma \otimes_A B &\xrightarrow{1 \otimes \Delta \otimes 1} B \otimes_A \Gamma \otimes_A \Gamma \otimes_A B \cong B \otimes_A \Gamma \otimes_A A \otimes_A \Gamma \otimes_A B \\ &\xrightarrow{1 \otimes 1 \otimes f \otimes 1 \otimes 1} B \otimes_A \Gamma \otimes_A B \otimes_A \Gamma \otimes_A B \cong (B \otimes_A \Gamma \otimes_A B) \otimes_B (B \otimes_A \Gamma \otimes_A B). \end{aligned}$$

We leave it to the reader to check that this does define a Hopf algebroid. We define $f_1: \Gamma \rightarrow \Gamma_f$ to be the composite

$$\Gamma \cong A \otimes_A \Gamma \otimes_A A \xrightarrow{f \otimes 1 \otimes f} B \otimes_A \Gamma \otimes_A B.$$

We leave it to the reader to check that this defines a map of Hopf algebroids, and also to check our uniqueness claims.

We therefore have the following corollary.

Corollary 5.6. *Suppose $f = (f_0, f_1): (A, \Gamma) \rightarrow (B, \Sigma)$ is a map of Hopf algebroids. Then $f^*: (\text{Spec } B, \text{Spec } \Sigma) \rightarrow (\text{Spec } A, \text{Spec } \Gamma)$ is an internal equivalence in the flat topology if and only if (B, Σ) is isomorphic over (A, Γ) to (B, Γ_{f_0}) and there is a ring map $g: B \otimes_A \Gamma \rightarrow C$ such that $g(f_0 \otimes \eta_R)$ exhibits C as a faithfully flat extension of A .*

The conditions in Corollary 5.6 have appeared before, in [HS99, Theorem 3.3] and in [Hop95]. Of course, in the situation of Corollary 5.6, Theorem 4.5 gives us an equivalence of categories between (A, Γ) -comodules and (B, Γ_f) -comodules. This equivalence of categories takes an (A, Γ) -comodule M to $B \otimes_A M$.

6. FORMAL GROUPS

In this section, we apply Corollary 5.6 and the theory of formal group laws to prove Theorem E. We also recover the change of rings theorems of Miller-Ravenel [MR77] and Hovey-Sadofsky [HS99].

This section requires familiarity with formal group laws and how they are used in algebraic topology. A good source for this material is [Rav86], especially Appendix 2 for formal group laws and Chapter 4 for their use in algebraic topology.

Fix a prime p for use throughout this section. Recall that (BP_*, BP_*BP) is the universal Hopf algebroid for p -typical formal group laws. Here $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, and $BP_*BP = BP_*[t_1, t_2, \dots]$; see [Rav86, Section 4.1]. The fact that (BP_*, BP_*BP) is universal means that a p -typical formal group law over a ring R is equivalent to a ring homomorphism $BP_* \rightarrow R$, and a strict isomorphism of p -typical formal group laws over R is equivalent to a ring homomorphism $BP_*BP \rightarrow R$. In case R is graded, let us call a p -typical formal group law over R *homogeneous* if its classifying map $BP_* \rightarrow R$ preserves the grading. (An example of a non-homogeneous formal group law is the formal group law over \mathbb{F}_p whose classifying map takes v_i to 0 for $i \neq n$ and v_n to 1).

Recall also the invariant ideal $I_n = (p, v_1, \dots, v_{n-1})$. The element v_n is a primitive modulo I_n . This means that there is a Hopf algebroid

$$(A, \Gamma) = (v_n^{-1}BP_*/I_n, v_n^{-1}BP_*BP/I_n).$$

Definition 6.1. A p -typical formal group law over a ring R is said to have *strict height n* if its classifying map factors through $v_n^{-1}BP_*/I_n$.

Our application of Theorem 4.5 is then the following theorem, which is Theorem E of the introduction.

Theorem 6.2. Fix a prime p and an integer $n > 0$. Let (A, Γ) denote the Hopf algebroid $(v_n^{-1}BP_*/I_n, v_n^{-1}BP_*BP/I_n)$. Suppose B is a graded ring equipped with a homogeneous p -typical formal group law of strict height n , classified by $f: A \rightarrow B$. Then the functor that takes an (A, Γ) -comodule M to $B \otimes_A M$ defines an equivalence of categories from graded (A, Γ) -comodules to graded (B, Γ_f) -comodules.

Proof. Let $D = A \otimes_{\mathbb{F}_p[v_n, v_n^{-1}]} B$. Let $x: A \rightarrow D$ denote the ring homomorphism defined by $x(a) = a \otimes 1$, and let $y: B \rightarrow D$ denote the ring homomorphism defined by $y(b) = 1 \otimes b$. Then x and the composite yf induce two formal group laws F and G over D , both p -typical and of strict height n . Furthermore, $x(v_n) = yf(v_n)$. A result of Lazard, as modified by Strickland [HS99, Theorem 3.4], then implies that there is a faithfully flat graded ring extension $h: D \rightarrow C$ and a strict isomorphism from h_*G to h_*F . This strict isomorphism is represented by a graded ring homomorphism $\alpha: \Gamma \rightarrow C$. Let $g: B \rightarrow C$ be the composite hy . Since the domain of α is h_*G , $\alpha\eta_L = gf: A \rightarrow C$. This means that there is a well-defined map

$$g \otimes \alpha: B \otimes_A \Gamma \xrightarrow{g \otimes \alpha} C_{gf} \otimes_A \alpha\eta_L C \xrightarrow{\mu} C.$$

Furthermore, $(g \otimes \alpha) \circ (f \otimes \eta_R)$ represents the codomain of α , so is hx . We know already that h is a faithfully flat ring extension, and we claim that x is also a faithfully flat ring extension. Indeed, since $\mathbb{F}_p[v_n, v_n^{-1}]$ is a graded field, B is a free $\mathbb{F}_p[v_n, v_n^{-1}]$ -module, and so x makes D into a free A -module. Corollary 5.6 and Theorem 4.5 complete the proof. \square

In particular, we can take $B = E(m)_*/I_n$, where $m \geq n$ and $E(m)$ is the Landweber exact Johnson-Wilson homology theory introduced in [JW75]. This leads to the following corollary.

Corollary 6.3. *Let p be a prime and $m \geq n > 0$ be integers. Then the functor that takes M to $E(m)_* \otimes_{BP_*} M$ defines an equivalence of categories*

$$(v_n^{-1}BP_*/I_n, v_n^{-1}BP_*BP/I_n)\text{-comodules} \\ \rightarrow (v_n^{-1}E(m)_*/I_n, v_n^{-1}E(m)_*E(m)/I_n)\text{-comodules}.$$

Using the method of [MR77], we then get the following change of rings theorem, which is Theorem F of the introduction.

Theorem 6.4. *Let p be a prime and $m \geq n > 0$ be integers. Suppose M and N are BP_*BP -comodules such that v_n acts isomorphically on N . If either M is finitely presented, or if $N = v_n^{-1}N'$ where N' is finitely presented and I_n -nilpotent, then*

$$Ext_{BP_*BP}^{**}(M, N) \cong Ext_{E(m)_*E(m)}^{**}(E(m)_* \otimes_{BP_*} M, E(m)_* \otimes_{BP_*} N).$$

Note that, when $M = BP_*$, this is the Hovey-Sadofsky change of rings theorem [HS99, Theorem 3.1]. When $m = n$ and $M = BP_*$, we get the Miller-Ravenel change of rings theorem [MR77, Theorem 3.10].

Proof. By Lemma 3.11 of [MR77], N is the direct limit of comodules $v_n^{-1}N'$, where N' is finitely presented and I_n -nilpotent. Since we are assuming either that M is finitely presented or that $N = v_n^{-1}N'$, in either case we may as well take $N = v_n^{-1}N'$. Then Lemma 3.12 of [MR77] reduces us to the case $N = v_n^{-1}BP_*/I_n$. In this case, one can check using the cobar resolution (as in [MR77, Proposition 1.3]) that we have canonical isomorphisms

$$Ext_{BP_*BP}^{**}(M, N) \cong Ext_{v_n^{-1}BP_*BP/I_n}^{**}(v_n^{-1}M/I_n, N)$$

and

$$Ext_{E(m)_*E(m)}^{**}(E(m)_* \otimes_{BP_*} M, E(m)_* \otimes_{BP_*} N) \cong \\ Ext_{v_n^{-1}E(m)_*E(m)/I_n}^{**}(E(m)_* \otimes_{BP_*} v_n^{-1}M/I_n, E(m)_* \otimes_{BP_*} N)$$

Now Corollary 6.3 implies that

$$Ext_{v_n^{-1}BP_*BP/I_n}^{**}(v_n^{-1}M/I_n, N) \cong \\ Ext_{v_n^{-1}E(m)_*E(m)/I_n}^{**}(E(m)_* \otimes_{BP_*} v_n^{-1}M/I_n, E(m)_* \otimes_{BP_*} N).$$

This completes the proof. \square

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